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DYNAMICS OF ACCUMULATION OF NEGATIVE-BUOYANCY BLOWOUTS INTO THE ATMOSPHERE IN WINDLESS WEATHER

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The problem and the corresponding mathematical formulation which describe the accumulation of negative-buoyancy blowouts on a horizontal surface with account for the resistance of flora and the ground are presented. Self-similar and approximate analytical solutions are constructed in the presence of flora and in the absence of it when the force of resistance on the source side of the ground is specified by two different expressions.

Introduction. The problem of description of the processes of propagation and accumulation of heavy gas blowouts into the atmosphere is especially pressing for cities with a developed chemical industry. The formation of the level of contamination of the air basin of these cities is affected by both the quantitative and qualitative composition, the intensity of blowouts, and meteorological conditions. In what follows, a gas mixture and a mixture of a gas with solid or liquid particles of negative buoyancy is termed smog (the density of the smog is higher than the density of atmospheric air). As a result, smog spreads along the underlying ground. The leading role in the propagation of smog is played by buoyancy forces. In the present work, the propagation of smog is studied on the basis of a theoretical model constructed similarly to the theory of shallow water [1]. In the mathematical description of this process, we take the following assumptions: the smog propagates in windless weather; the ground relief is a smooth horizontal surface.

Basic Equations. Let there be a point source of smog. By this we will mean an object with transverse and longitudinal dimensions of the same order. To simplify a mathematical formulation of the problems we will neglect the linear dimensions of blowout sources. This assumption means that in what follows we will be interested in distances which are much larger than the characteristic dimensions of smog sources. Under the assumptions made, the equations of conservation of mass and momenta in the quasi-one-dimensional approximation have the form [2]

$$\frac{\partial h}{\partial t} + \frac{1}{r}\frac{\partial}{\partial r}(rhu) = 0\,, \qquad (1)$$

$$\frac{du}{dt} = -g'\frac{\partial h}{\partial r} - \frac{\tau_G + \tau_A + \tau_F}{h}\,,$$
$$\frac{d}{dt} = \frac{\partial}{\partial t} + u\frac{\partial}{\partial r}\,; \quad g' = g\frac{\rho - \rho_A}{\rho}\,. \qquad (2)$$

We note that (1) is written with the mass exchange between the smog layer and the atmospheric air above the smog layer ($z > h$) and the volume fraction of surface objects (trees, houses) being neglected. In Eq. (2), the effect of friction is allowed for by introduction of the forces of resistance which depend on velocity both linearly and according to the square law and which are related to h in a certain manner which will be described below.

Two situations are possible in spreading of smog: the smog height is lower than the level of surface objects (force of resistance τ_F prevails); the smog height is much higher than surface objects (force τ_G prevails).

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Dynamics of Accumulation of Smog in the Presence of Flora. We consider a situation which corresponds to the propagation of smog, for example, in the forest where the force τ_F prevails. In this case, we assume that the thickness of the smog is lower than the level of trees. We can consider two limiting cases, namely: thick flora where the linear law of resistance can be taken (by analogy with the theory of underground hydraulics) and thinly growing flora. In the second case, we use the square law of resistance. Then, in accordance with what was stated above, we can write

$$f=\frac{u}{t_u}\,,\quad f=\frac{u^2}{r_u}\quad \left(f=\frac{\tau_F}{h}\right). \tag{3}$$

In particular, if we take Newton's formula for resistance, then in the case of the square law of resistance we have

$$r_u=\chi/nd\,.$$

An analysis shows that inertia effects (determined by the terms on the right-hand side of Eq. (2)) are usually substantial at the initial stage which satisfies the conditions (1) $t_* \sim t_u$, $r_* \sim u_* t_*/2$ for the linear law of resistance and (2) $t_* \sim r_u/u_*$, $r_* \sim r_u/2$ for the square law of resistance, where u_* is the maximum smog velocity, which is similar to sound "choking" in gas dynamics and is bounded by the value $u_* = \sqrt{g' h_*}$.

It is apparent that in spreading of the smog, the situation where the effect of the gravity force counterbalances the force of resistance (inertia forces are negligible) is of greatest interest. Then, on the basis of the equations of conservation of mass and momenta, by neglecting the terms on the left-hand side of Eq. (2) we obtain

$$\frac{\partial h}{\partial t}=\frac{g'_{(1)}}{r}\frac{\partial}{\partial r}\left(rh\frac{\partial h}{\partial r}\right),\quad u=-g'_{(1)}\frac{\partial h}{\partial r}\quad (g'_{(1)}=g' t_u)\,, \tag{4}$$

$$\frac{\partial h}{\partial t}=-\frac{\sqrt{g'_{(2)}}}{r}\frac{\partial}{\partial r}\left(rh\sqrt{\frac{\partial h}{\partial r}}\right),\quad u=\sqrt{-g'_{(2)}\frac{\partial h}{\partial r}}\quad (g'_{(2)}=g' r_u) \tag{5}$$

for the cases of linear and square laws of resistance. We note that (4) is the Boussinesq equation widely used in underground hydraulics [3].

Let there be no smog at the initial instant of time, and at a certain instant $t = 0$ a point source with a constant power ($Q(t)$ = const) begins to function. Then the initial and boundary conditions have the form

$$h=0\quad (t=0\,,\ r>0)\,,\quad (2\pi\, rhu)_{r_c}=Q\quad (t>0\,,\ r_c\to 0)\,. \tag{6}$$

In the case of the linear law of resistance, this problem has a self-similar solution. We introduce the dimensionless height of the smog and the self-similar coordinate

$$H=h/h_{(1)}\,,\quad \xi=r/\sqrt{\eta_{(1)}t}\,,\quad \left(h_{(1)}=\sqrt{Q/g'_{(1)}}\,,\quad \eta_{(1)}=\sqrt{Qg'_{(1)}}\right). \tag{7}$$

Then Eq. (4) can be transformed as

$$-\frac{\xi}{2}\frac{dH}{d\xi}=\frac{1}{\xi}\frac{d}{d\xi}\left(H\xi\frac{dH}{d\xi}\right). \tag{8}$$

In this case, initial and boundary conditions (6) have the form

$$H(\infty)=0\,,\quad 2\pi\xi_c\left(H\frac{dH}{d\xi}\right)_{\xi_c}=-1\quad (\xi_c\to 0)\,. \tag{9}$$

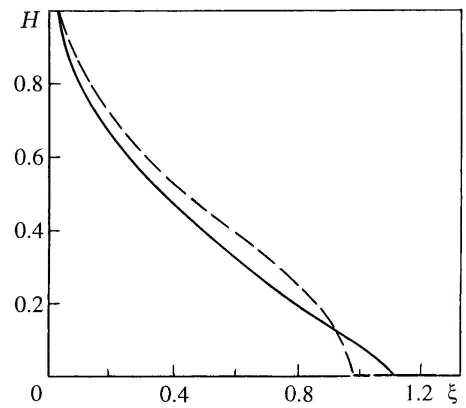


Fig. 1. Distribution of the dimensionless height of the smog in self-similar variables for the case of thick flora at a constant power of the smog source.

Moreover, the approximate analytical solution of Eq. (4) can be obtained by the method of successive substitution of steady states (SSSS) [3]. As is known, for many problems associated with solution of the nonlinear equation of heat conduction this method can be used to determine approximate analytical solutions with an accuracy necessary for many practical problems. Its essence is as follows. It is assumed that the distribution of parameters which describe the dynamics of the process along the coordinate at any instant of time (in our case, the smog height) is similar to a steady-state process. In other words, using steady-state solutions one constucts splines (analytical formulas) to obtain approximate solutions of the initial problem. The equation for this distribution (for determination of splines) is obtained on the basis of (4) and (5) by equating their left-hand sides to zero ($\partial h/\partial t = 0$). Then, the solution, with account for the condition $h = 0$ at $r = l(t)$, can be written in terms of an unknown function $l(t)$. In this case, the equation for the law of motion of the front boundary of the smog $l(t)$ can be obtained on the basis of the equation of balance of mass in integral form, and the solution for this problem has the form

$$h = \sqrt{\frac{Q}{\pi g'_{(1)}} \ln\left(\frac{l(t)}{r}\right)}\,, \quad l(t) = 0.97\,(Q g'_{(1)} t^2)^{1/4} \quad (0 < r \le l(t))\,, \quad h = 0 \quad (r > l(t))\,. \tag{10}$$

We represent the solution (10) in the reduced variables (7) as

$$H = \left(\frac{1}{\pi} \ln\left[\frac{\xi_0}{\xi}\right]\right)^{1/2}\,, \quad \xi_0 = 0.97 \quad (0 \le \xi \le \xi_0)\,, \quad H = 0 \quad (\xi > \xi_0)\,. \tag{11}$$

Figure 1 presents the distributions of the smog layer for the linear law of resistance (thick flora). The solid line corresponds to the numerical solution of Eq. (8), while the dashed line corresponds to the solution (11) with account for (9). In this case, for laws of motion of the leading front of the smog which correspond to the self-similar and approximate solutions we have

$$l(t) = 1.11\,(\eta_{(1)} t)^{1/2} \quad \text{and} \quad l(t) = 0.97\,(\eta_{(1)} t)^{1/2}\,.$$

In the case of thinly growing flora, the solution obtained by the SSSS method with conditions (6) has the form

$$h = \sqrt[3]{\frac{3Q^2}{4\pi^2 g'_{(2)}}\left(\frac{1}{r} - \frac{1}{l(t)}\right)}\,, \quad l(t) = 1.07\,(g'_{(2)} Q t^3)^{1/5} \quad (0 < r \le l(t))\,, \quad h = 0 \quad (r > l(t))\,. \tag{12}$$

It follows for the solutions (10) and (12) and for the numerical solution of Eq. (8) that the smog height tends to infinity when $r \to 0$. This singularity is associated with the fact that the linear dimensions of the smog source are disregarded in the formulation of the problem. Moreover, we note that there is a fundamental difference in the character of distribution of the smog between these two situations described by the solutions (10) and (12). Whereas in the case of the linear law of resistance for any fixed coordinate r the smog thickness increases without bound ($h \to \infty$ when $t \to \infty$), in the case of the square law when $t \to \infty$ there is a limiting height for each value of r, which is determined from the expression

$$h^{(\mathrm{m})} = \sqrt[3]{\frac{3Q^2}{4\pi^2 g'_{(2)} r}} \, .$$

Consequently, irrespective of the time of operation of the source the smog height will not exceed the value of $h^{(\mathrm{m})}$.

For the square law of resistance the problem has a self-similar solution if the dependence of the source power on time has the form

$$Q = qt^{1/3} \, . \tag{13}$$

We introduce the dimensionless height and the self-similar variable

$$H = h/h_{(2)} \, , \quad \xi = \frac{r}{(\eta_{(2)} t)^{2/3}} \, (h_{(2)} = (q^3/g'^2_{(2)})^{1/5} \, , \quad \eta_{(2)} = (g'_{(2)} q)^{3/10}) \, . \tag{14}$$

Then Eq. (15) in the self-similar variables (14) is represented as

$$-\frac{2}{3} \xi \frac{dH}{d\xi} + \frac{1}{\xi} \frac{d}{d\xi} \left(\xi H \sqrt{-\frac{dH}{d\xi}} \right) = 0 \, . \tag{15}$$

In this case, initial and boundary conditions (6) and (13) in the self-similar variables (14) can be written as follows:

$$H(\infty) = 0 \, , \quad 2\pi \xi_{\mathrm{c}} H \sqrt{-\left(\frac{dH}{d\xi}\right)_{\xi_{\mathrm{c}}}} = 1 \quad (\xi_{\mathrm{c}} \to 0) \, . \tag{16}$$

Numerical solution of Eq. (15) with account for (16) yielded $\xi_0 = 1.025$. Then the expression for the law of motion of the leading front of the smog has the form $l(t) = 1.025(\eta_{(2)} t)^{1/2}$. The dependence of the dimensionless height of the smog H on the self-similar coordinate ξ is similar to that presented in Fig. 1.

An approximate analytical solution of Eq. (5) with conditions (6) and (13) has the form

$$h = \left(\frac{3q^2 t^{2/3}}{4\pi^2 g'_{(2)}} \left(\frac{1}{r} - \frac{1}{l(t)} \right) \right)^{1/3} , \quad l(t) = 0.81 \, (q g'_{(2)} t^{10/3})^{1/5} \quad (0 < r \le l(t)) \, , \quad h = 0 \quad (r > l(t)) \, , \tag{17}$$

and in the dimensionless self-similar variables (14) it can be written as

$$H = \left[\frac{3}{4\pi^2} \left(\frac{1}{\xi} - \frac{1}{\xi_0} \right) \right]^{1/3} l(t) = 0.81 \, (\eta_{(2)} t)^{2/3} \, , \quad \xi_0 = 0.81 \quad (0 < \xi \le \xi_0) \, , \quad H = 0 \quad (\xi > \xi_0) \, . \tag{18}$$

Dynamics of Accumulation of Smog in the Absence of Flora. We consider the situation which corresponds to the propagation of smog over a horizontal surface in the absence of trees ($\tau_{\mathrm{F}} = 0$), e.g., in a field. Moreover, we neglect the force of resistance on the source side of the atmosphere ($\tau_{\mathrm{G}} >> \tau_{\mathrm{A}}$). The force of resistance on the source side of the ground is specified as follows:

$$\tau_G = \lambda u^2 . \tag{19}$$

We consider two approaches to specification of the coefficient λ. In the first case, we take λ = const. On the basis of the data of [4–7], we have $\lambda = (1.4–2.5)\cdot 10^{-3}$ for a smooth field, $\lambda = (1.5–1.7)\cdot 10^{-3}$ for a smooth surface, and $\lambda = 1.42\cdot 10^{-3}$ for a smooth snow-covered icy field.

Then, on the basis of Eqs. (1) and (2) with account for (19) and with the inertia effects being disregarded, we obtain

$$\frac{\partial h}{\partial t} = -\frac{1}{r}\frac{\partial}{\partial r}\left(rh\sqrt{-g'_{(\lambda)}h\frac{\partial h}{\partial r}}\right), \quad u = \sqrt{-g'_{(\lambda)}h\frac{\partial h}{\partial r}} \quad \left(g'_{(\lambda)} = \frac{g'}{\lambda}\right). \tag{20}$$

The equation which coincides in form with (20) has been obtained in [8] for a description of large shallow water areas.

An analytical solution corresponding to conditions (6) where the power of a smog source is constant can be written as

$$h = \left(\frac{Q^2}{\pi^2 g'_{(\lambda)}}\left(\frac{1}{r} - \frac{1}{l}\right)\right)^{1/4}, \quad l(t) = 0.8\,(Q^2 g'_{(\lambda)} t)^{4/7} \quad (0 < r \le l(t)), \quad h = 0 \quad (r > l(t)) . \tag{21}$$

Let the earth's surface be clean at the initial instant of time and a smog source with a power dependent on time according to (13) begin to function after a certain period. In this case, the problem also has a self-similar solution. We introduce the dimensionless height and the self-similar variable in the following manner:

$$H = h/h_{(3)}, \quad \xi = \frac{r}{(\eta_{(\lambda)}t)^{2/3}} \quad \left(h_{(3)} = \left(\frac{q^3}{g'^2_{(\lambda)}}\right)^{1/7}, \quad \eta_{(\lambda)} = (g'^3_{(\lambda)}q^6)^{1/14}\right). \tag{22}$$

Then Eq. (20) in the variables (22) takes on the form

$$-\frac{2}{3}\xi\frac{dH}{d\xi} + \frac{1}{\xi}\frac{d}{d\xi}\left(\xi H\sqrt{-\frac{dH}{d\xi}}\right) = 0 . \tag{23}$$

Initial and boundary conditions (6) and (13) in the self-similar variables (22) can be written as

$$H(\infty) = 0, \quad 2\pi\xi_c H\sqrt{-H\left(\frac{dH}{d\xi}\right)_{\xi_c}} = 1 \quad (\xi_c \to 0) . \tag{24}$$

Numerical solution of Eq. (23) with account for (24) yielded $\xi_0 = 0.75$. Then the expression for the law of motion of the leading front of the smog has the form $l(t) = 0.75\,(\eta_{(\lambda)}t)^{2/3}$. The dependence of $H(\xi)$ on the self-similar coordinate ξ is similar to that presented in Fig. 1 (solid curve).

Using the SSSS method, we obtain the following analytical solution for this problem:

$$h = \left(\frac{q^2 t^{2/3}}{\pi^2 g'_{(\lambda)}}\left(\frac{1}{r} - \frac{1}{l}\right)\right)^{1/4}, \quad l(t) = 0.69\left(g'^{1/2}_{(\lambda)} q t^{7/3}\right)^{2/7} \quad (0 < r \le l(t)), \quad h = 0 \quad (r > l(t)) .$$

In the self-similar variables (22) it has the form

$$H = \left[\frac{1}{\pi^2}\left(\frac{1}{\xi} - \frac{1}{\xi_0}\right)\right]^{1/4}, \quad l(t) = \xi_0\,(\eta_{(\lambda)}t)^{2/3}, \quad \xi_0 = 0.69 \quad (0 < \xi \le \xi_0), \quad H = 0 \quad (\xi > \xi_0) . \tag{25}$$

In the second case, we take Manning's law for the law of resistance [9, 10]. Then we can write the following for the coefficient λ:

$$\lambda = \left(\frac{h_{**}}{h}\right)^{1/3} . \tag{26}$$

Values for h_{**} can be estimated on the basis of the data presented in [11]. For this case we use Manning's law of resistance (26) for the coefficient of resistance and on the basis of Eqs. (1) and (2) with the inertia effects being neglected we obtain

$$\frac{\partial h}{\partial t} = -\frac{1}{r}\frac{\partial}{\partial r}\left(rh\sqrt{-\tilde{g}'_{(\lambda)}h^{4/3}\frac{\partial h}{\partial r}}\right), \quad u = \sqrt{-\tilde{g}'_{(\lambda)}h^{4/3}\frac{\partial h}{\partial r}} \quad \left(\tilde{g}'_{(\lambda)} = g'/h_{**}\right). \tag{27}$$

We represent the analytical solution at a constant power of the source (conditions (6)) as

$$h = \left(\frac{13}{12}\frac{Q^2}{\pi^2\tilde{g}'_{(\lambda)}}\left(\frac{1}{r} - \frac{1}{l}\right)\right)^{3/13} , \quad l\,(t) = 0.77\left(Q^7\,(\tilde{g}'_{(\lambda)})^3\,t^{13}\right)^{1/23} \quad (0 < r \le l\,(t)) , \quad h = 0 \quad (r > l\,(t)) .$$

The self-similar solution of the problem exists if the dependence of the power of the smog source on time has the form (13). We introduce the dimensionless height and the self-similar variable in the following manner:

$$H = h/\tilde{h}_{(\lambda)} , \quad \xi = \frac{r}{(\tilde{\eta}_{(\lambda)}t)^{2/3}} , \quad \tilde{h}_{(\lambda)} = \left(\frac{q^3}{\tilde{g}'^2_{(\lambda)}}\right)^{1/9} , \quad \tilde{\eta}_{(\lambda)} = (q^3\tilde{g}'_{(\lambda)})^{1/6} . \tag{28}$$

Then Eq. (27) in the variables (28) is written in the form

$$-\frac{2}{3}\,\xi\,\frac{dH}{d\xi} + \frac{1}{\xi}\frac{d}{d\xi}\left(\xi H\sqrt{-H^{4/3}\frac{dH}{d\xi}}\right) = 0 . \tag{29}$$

We represent initial and boundary conditions (6) and (13) in the self-similar variables (28) as

$$H\,(\infty) = 0 , \quad 2\pi\xi_c H\sqrt{-H^{4/3}\left(\frac{dH}{d\xi}\right)_{\xi_c}} = 1 \quad (\xi_c \to 0) . \tag{30}$$

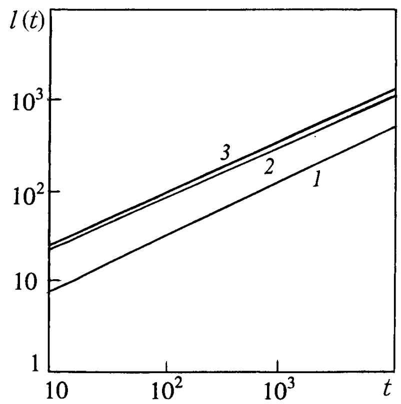
Numerical solution of Eq. (29) with account for (30) yielded $\xi_0 = 0.71$. Then the expression for the law of motion of the leading front of the smog has the form

$$l\,(t) = 0.71\,(\tilde{\eta}_{(\lambda)}t)^{2/3} .$$

We represent the approximate solution of Eq. (27) obtained by the SSSS method with initial and boundary conditions (6) and (13) as

$$h = \left(\frac{13}{12}\frac{q^2t^{2/3}}{\pi^2\tilde{g}'_{(\lambda)}}\left(\frac{1}{r} - \frac{1}{l}\right)\right)^{3/13} , \quad l\,(t) = 0.67\left(\tilde{g}'^{1/6}_{(\lambda)}\,q^{1/2}t\right)^{2/3} \quad (0 < r \le l\,(t)) , \quad h = 0 \quad (r > l\,(t)) . \tag{31}$$

In the self-similar variables (28) it can be written as



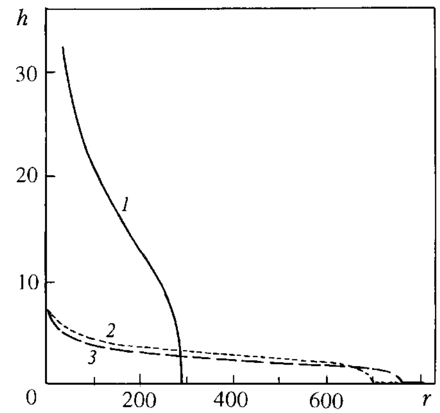


Fig. 2. Dependence of the coordinate of the front boundary of the smog layer on time: 1) predominance of resistance on the source side of flora; 2 and 3) absence of flora for two approaches to specification of the coefficient of resistance, respectively.

Fig. 3. Distribution of the smog layer for different laws of resistance: 1) thinly growing flora; 2 and 3) absence of flora [forces of resistance are specified by Manning's law and at λ = const, respectively] for a time of 1 h.

$$H = \left[\frac{13}{12}\left(\frac{1}{\xi} - \frac{1}{\xi_0}\right)\right]^{3/13}, \quad l(t) = \xi_0 \, (\tilde{\eta}_{(\lambda)} t)^{2/3}, \quad \xi_0 = 0.67 \quad (0 < \xi \le \xi_0), \quad H = 0 \quad (\xi > \xi_0). \tag{32}$$

Figure 2 and 3 present graphs which illustrate certain qualitative and quantitative features of evolution of the smog layer for different types of the square law of resistance determined by formulas (3) and (19) at the same values of the blowout intensity Q = 1000 m^3/sec and smog density ρ = 1.3013 kg/m^3. For the other parameters, which determine the state of the atmosphere and the force of resistance, we took ρ_A = 1.3 kg/m^3, r_u = 5 m, $\lambda = 2.5 \cdot 10^{-3}$, and $h_{**} = 2.3 \cdot 10^{-7}$ m (g' = 0.01 m/sec^2). We note that the value for r_u corresponds to the forest thickness n = 0.4 with a typical diameter of the trees of d = 0.5 m, while the reduced value for h_{**} is obtained on the basis of the tabular data from [11] for a smooth fluvial plain. It follows from the comparison of curves 2 and 3 (Figs. 2 and 3) that the two approaches to specification of the coefficient λ in (19) give close results. Furthermore, a strong influence of flora on the spreading of blowouts is seen from the curves presented.

Conclusions. On the basis of the model of quasi-one-dimensional theory that was taken, we have revealed certain qualitative and quantitative features of spreading of heavy mixtures (as compared to air) along the underlying ground for different laws for the force of resistance. The obtained numerical self-similar and approximate analytical solutions can be used to test computation algorithms by more complex mathematical models which allow for heat and mass transfer processes and multidimensional effects.

NOTATION

r, distance reckoned from the source, m; t, time, sec; u, velocity of the smog, m/sec; h, smog thickness, m; z, vertical coordinate along which h is measured, m; g, gravitational acceleration, m/sec^2; ρ and ρ_A, density of the smog and the atmospheric air, kg/m^3; τ_G, τ_A, and τ_F, reduced forces of resistance related to columns of smog with a unit base on the source side of the earth's surface, the atmospheric air, and surface objects distributed over the earth's surface, m^2/sec^2; t_u, empirical parameter, sec; d, typical diameter of the trees, m; n, number of trees per unit area (forest thickness), 1/m^2; χ, dimensionless coefficient ($\chi \sim 1$); h_*, certain characteristic thickness of the layer, m; Q, source power, m^3/sec; $l(t)$, coordinate of the front boundary of the smog layer, m; h_{**}, effective parameter responsible for the roughnesses of the earth's surface, m; λ, coefficient of resistance on the source side of the earth's surface. Subscripts: c, center; m, maximum; G, ground; A, atmosphere; F, flora.

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